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ITERATED $\bar{W}$ and T-GROUPOIDS

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The classical Eilenberg–MacLane classifying complex functor $\bar{W}$ for simplicial groups has been generalized by Duskin to a functor on bisimplicial objects. When restricted to the category of (simplicial) T-groupoids whose complex of objects is constant, it is part of the equivalence to the category of T-complexes. The construction of its inverse, a loop functor, necessitated the development of new proof techniques using recursively defined trees of horns. A consequence of this equivalence is that multiple groupoid structures may be systematically unfolded (by iterating $\bar{W}$) to form equivalent simplicial ones.

Introduction

In this companion paper to [14] we show that the category **Jgpd** of (simplicial) T-groupoids whose complex of objects is constant is equivalent to the category of T-complexes. We will follow the notation introduced there.

The classical Eilenberg–MacLane classifying complex functor $\bar{W}$ for simplicial groups has been generalized by Duskin to a functor on bisimplicial objects. When restricted to **Jgpd** it is part of the equivalence which turns out to be identical to one considered by Artin and Mazur [1], between **Jgpd** and the category of T-complexes. Its inverse G, a loop functor, associates new groupoid structures to a T-complex (Section 2). More explicitly, the pullback of Dec^1 over the 0-skeleton is endowed with canonical groupoid structures making it a T-groupoid. Now an essential feature of T-complexes (as noted by Brown) is that algebraic operations are determined by faces of particular simplices. The construction of G necessitates the introduction of new proof techniques using recursively defined trees of horns (cf. collapsing in [4]). Here, the vertices of the tree are horns Λ_i ('ith face missing') and the directed edges of the tree $\Lambda_i \to \Lambda_j$ $(i \neq j)$ are described by: a face of Λ_j is the ith face of a thin filler of Λ_i. It is the roots of these trees that produce simplices whose faces have desired properties. (See Section 3 for example.)

Since the equivalence between **Jgpd** and the category of T-complexes holds for the truncated versions with a dimension shift, we see that iterating $\bar{W}$ gives an equiva-

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lence between the category of ∞-groupoids and **Jgpd** (and hence T-complexes). So that, roughly speaking, multiple groupoid structures may be systematically unfolded to form equivalent simplicial ones. See also [12] for related work.

Thus in conjunction with [14] we have the following diagram of equivalences relative to any category $\mathscr{C}$ with finite limits, for $n\geq 1$:

$$\begin{tikzcd}[column sep=small] & (n\text{-groupoids}) \arrow[d, phantom, "\wr"] & \\ & (n-1)\text{-}\mathbf{Jgpd} \arrow[dl, shift left, "\bar{W}"] \arrow[dr, shift left, "N"] & \\ (n\text{-}T\text{-complexes}) \arrow[ur, shift left, "G"] & & (n\text{-crossed complexes}) \arrow[ul, shift left, "K"] \end{tikzcd}$$

We note that the equivalence between ∞-groupoids and crossed complexes was shown directly by Brown–Higgins [3]. Ashley's homotopy crossed complex functor [2] is the composition of a pair of these equivalences when $\mathscr{C}$ is the category of sets. The one-object case of this was first observed by Duskin [6]. In this framework, all of these equivalences are seen in terms of generalizations of classical functors definable in any category with finite (inverse) limits.

1. The functors $(m-1)$-Jgpd$(\mathbb{C}) \underset{G}{\overset{\bar{W}}{\rightleftarrows}} m$-$T$-$\mathbb{C}$

Duskin has defined a functor $\mathbb{C}^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \xrightarrow{\bar{W}} \mathbb{C}^{\Delta^{\mathrm{op}}}$ as follows:

$$\bar{W}(X..)_n=\{(a_{p,n-p})_{0\leq p\leq n}\mid d_0^H(a_{p,n-p})=d_{p+1}^V(a_{p+1,n-p-1}),\ 0\leq p\leq n-1\}.$$

Face and degeneracy operators $\bar{W}(X..)_n \underset{s_j}{\overset{d_j}{\rightleftarrows}} \bar{W}(X..)_{n-1}$ are given by

$$(d_j(a..))_p=\begin{cases} d_j^V(a_{p+1,n-p-1}) & \text{for } j\leq p,\\ d_{j-p}^H(a_{p,n-p}) & \text{for } j>p,\end{cases}$$

for $0\leq p\leq n-1$ and $0\leq j\leq n$;

$$(s_j(a..))_p=\begin{cases} s_j^V(a_{p-1,n-p}) & \text{for } j<p,\\ s_{j-p}^H(a_{p,n-p-1}) & \text{for } j\geq p,\end{cases}$$

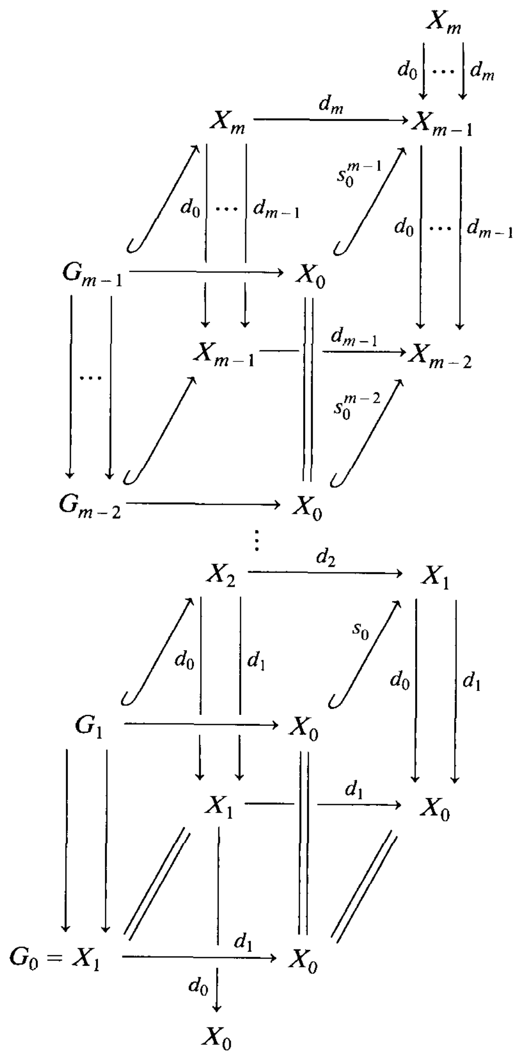
for $0\leq p\leq n$ and $0\leq j\leq n-1$.

He has also shown that $\bar{W}$ is a right adjoint, hence it preserves limits, in particular filtered systems of hypergroupoids. Thus $\bar{W}$ applied to $(m-1)$-**Jgpd**$(\mathbb{C})$ is a functor into m-T-$\mathbb{C}$.

We now define a functor $G: m\text{-}T\text{-}\mathbb{C}\to(m-1)$-**Jgpd**$(\mathbb{C})$. Let $X_n=(X._m)_n$ for $0\leq n\leq m$, where $X._1\hookrightarrow X._2\hookrightarrow\cdots\hookrightarrow X._m\in m\text{-}T\text{-}\mathbb{C}$, and let G_{p-1} be the pullback given by

$$\begin{tikzcd} G_{p-1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_p \arrow[d, "d_p"] \\ X_0 \arrow[r, hook, "s_0^{p-1}"'] & X_{p-1} \end{tikzcd} \quad \text{for } 1 \le p \le m. \tag{1}$$

Then in $\mathbb{C}$ we have the following diagram:



(2)

Let $G_{\cdot q} = \operatorname{cosk}^q(G_q \mathrel{\substack{\to \\ \vdots \\ \to}} G_{q-1} \cdots G_1 \rightrightarrows G_0)$ for $1 \le q \le m-1$. Consider the following diagram:

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & G_{n-1} \arrow[r] \arrow[d, hook] & X_0 \arrow[d, hook, "s_0^{n-1}"] \\
\Lambda^n \arrow[r] & X_n \arrow[r, "d_n"'] & X_{n-1}
\end{tikzcd}
\tag{3}
$$

Since both squares are pullbacks, the outside rectangle is also a pullback. Hence the filtration $X._1 \hookrightarrow X._2 \hookrightarrow \cdots \hookrightarrow X._m$ induces a filtration $G._1 \hookrightarrow G._2 \hookrightarrow \cdots \hookrightarrow G._{(m-1)}$. We will show that $G_n \rightrightarrows X_0$ are groupoids where $\mathrm{dom} = d_1 d_0^n$ and $\mathrm{cod} = d_0^{n+1}$ and that the induced $d_i : G_n \to G_{n-1}$ are functors. So then $G(X._1 \hookrightarrow X._2 \hookrightarrow \cdots \hookrightarrow X._m) = G._1 \hookrightarrow G._2 \hookrightarrow \cdots \hookrightarrow G._{(m-1)} \in (m-1)$-**Jgpd** (with object complex X_0).

Morphisms of m-T-$\mathbb{C}$ induce maps in $\mathbb{C}$ between the pullbacks G_n. These result in morphisms of $(m-1)$-**Jgpd**. So G is a functor.

Suppose $(X, T) \in T$-**Set**.

For $n \geq 1$, let $G_{n-1} = \{x \in X_n \mid d_n x = s_0^{n-1} a, \text{ where } a \in X_0\}$. We will show that

$$
G_{n-1} \underset{\mathrm{dom} = d_1 d_0^{n-1}}{\overset{\mathrm{cod} = d_0^n}{\rightrightarrows}} X_0
$$

are groupoids and $d_j : G_n \to G_{n-1}$ are functors for $0 \leq j \leq n$.

The constructions involve trees whose vertices are horns Λ_i ('ith fact missing') and whose directed edges $\Lambda_i \to \Lambda_j$ $(i \neq j)$ are described by: a face of Λ_j is the ith face of a thin filler of Λ_i.

2. The multiplication

Proposition 2.0. *Suppose* $(X, T) \in T$-**Set**,

$$
x, y \in X_n, \quad n \geq 1, \qquad d_0^n x = d_1 d_0^{n-1} y,
$$

and

$$
d_n y = s_0^{n-1} b, \quad b \in X_0.
$$

Then for $1 \leq j \leq n$ *there exists a unique* $T_{n+1}^j \in T_{n+1} \subset X_{n+1}$ *such that*

$n = 1$:

$$
d_0 T_2^1[x, y] = y, \qquad d_2 T_2^1[x, y] = x;
$$

$n \geq 2$, $j = 1$:

$$
d_i T_{n+1}^1[x, y] = \begin{cases} y, & i = 0, \\ T_n^1[d_{n-1} x, d_{i-1} y], & 1 < i \leq n+1; \end{cases}
$$

$1 < j < n$:

$$
d_i T_{n+1}^j[x, y] = \begin{cases} T_n^{j-1}[d_i x, d_i y], & 0 \leq i < j-1, \\ d_{j-1} T_{n+1}^{j-1}[x, y], & i = j-1, \\ T_n^j[d_{n-1} x, d_{i-1} y], & j < i \leq n+1; \end{cases}
$$

$j=n$:

$$d_i T^n_{n+1}[x,y] = \begin{cases} T^{n-1}_n[d_i x, d_i y], & 0 \le i < n-1, \\ d_{n-1} T^{n-1}_{n+1}[x,y], & i = n-1, \\ x, & i = n+1. \end{cases} \qquad \square$$

The proof is by a long induction on n.

Observe that $T^1_{n+1}[x,y]$ is the thin filler of the horn

$$(y, \underset{1}{-}, T^1_n[x_{n-1}, y_1], \ldots, T^1_n[x_{n-1}, y_n]),$$

$T^j_{n+1}[x,y]$ is the thin filler of the horn

$$(T^{j-1}_n[x_0, y_0], \ldots, T^{j-1}_n[x_{j-2}, y_{j-2}], d_{j-1} T^{j-1}_{n+1}[x,y], \underset{j}{-}, T^j_n[x_{n-1}, y_j], \ldots, T^j_n[x_{n-1}, y_n]),$$

and $T^n_{n+1}[x,y]$ is the thin filler of the horn

$$(T^{n-1}_n[x_0, y_0], \ldots, T^{n-1}_n[x_{n-2}, y_{n-2}], d_{n-1} T^{n-1}_{n+1}[x,y], \underset{n}{-}, x).$$

When $d_n x = s_0^{n-1} a$, $a \in X_0$, as well.

Define $xy = d_n T^n_{n+1}[x,y]$.

Notation. Let

$$C^2_n = \{(x,y) \in X^2_n \mid d_0^n x = d_1 d_0^{n-1} y \text{ and } d_n y = s_0^{n-1} b,\ b \in X_0\}$$

and

$$C^3_n = \{(x,y,w) \in X^3_n \mid d_0^n x = d_1 d_0^{n-1} y \text{ and } d_n y = s_0^{n-1} b,\ b \in X_0 \text{ and } d_0^n y = d_1 d_0^{n-1} w \text{ and } d_n w = s_0^{n-1} c,\ c \in X_0\}.$$

Remark. Observe xy is defined for $(x,y) \in C^2_n$.

Lemma 2.1. *For $n \ge 2$, $(x,y) \in C^2_n$ we have:*

(i) $(x_{n-1}, y_{i-1}) \in C^2_{n-1}$, $i \le n+1$,

(ii) $(x_i, y_i) \in C^2_{n-1}$, $i < n-1$. $\square$

Thus the inductive hypothesis of Proposition 2.0 is well defined.

Recall, when $(x,y) \in C^2_n$ $(n \ge 1)$ with $d_n x = s_0^{n-1} a$ where $a \in X_0$, we defined $xy = d_n T^n_{n+1}[x,y] \in X_n$.

Proposition 2.2.

$$(xy)_i = x_i y_i, \quad 0 \le i \le n-1, \qquad (xy)_n = x_n.$$

If $x, y \in T_n$, then $xy \in T_n$.

Proof. $0 \le i < n-1$.

$$(xy)_i = d_i d_n T^n_{n+1}[x,y] = d_{n-1} d_i T^n_{n+1}[x,y] = d_{n-1} T^{n-1}_n[x_i, y_i] = x_i y_i.$$

$i=n-1$.

$$\begin{aligned}(xy)_{n-1} &= d_{n-1}d_nT^n_{n+1}[x,y] = d_{n-1}d_{n-1}T^n_{n+1}[x,y] = d_{n-1}d_{n-1}T^{n-1}_{n+1}[x,y]\\ &= d_{n-1}d_nT^{n-1}_{n+1}[x,y] = d_{n-1}T^{n-1}_n[x_{n-1},y_{n-1}] = x_{n-1}y_{n-1};\\ (xy)_n &= d_nd_nT^n_{n+1}[x,y] = d_nd_{n+1}T^n_{n+1}[x,y] = x_n. \qquad \square\end{aligned}$$

Corollary 2.3. *If* $(x,y)\in C_n^2$ *with* $d_ny=s_0^{n-1}b$ *and* $d_nx=s_0^{n-1}a$ *where* $a,b\in X_0$, *then* $s_j(xy)=s_jx\,s_jy$ *for* $0\le j\le n-1$. $\square$

So the face and degeneracy operators are functors.

3. Associativity

Suppose $(x,y,w)\in C_n^3$, we will construct trees of horns of the following form, by induction on n (see Fig. 1). The vertical uprights will be called *towers* whose *bases* form the diagonal slant.

The root of the tree will have properties that enable us to show associativity.

Scholium 3.0. Suppose $(X,T)\in T$-**Set** and $0<p<q<n+1$.

Suppose $x_k\in X_n$ where $0\le k\le n+1$ and $k\ne p,q$ satisfy $d_ix_j=d_{j-1}x_i$ for $i<j$.

Define $H\in\Lambda^q_{n-1}$ by

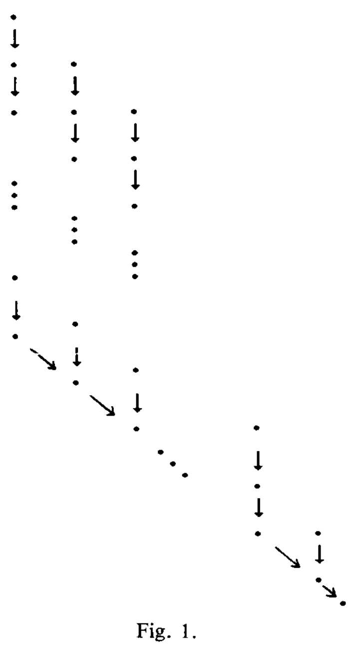


Fig. 1.

$$d_i H = \begin{cases} d_{p-1}x_i, & 0 \le i \le p-1, \\ d_p x_{i+1}, & p \le i \le n,\ i \ne q, \end{cases}$$

and let $F \in T_n$ be its unique thin filler. Then $L \in \Lambda_n^q$, where for $i \ne q$,

$$d_i L = \begin{cases} F, & i = p, \\ x_i, & \text{otherwise.} \end{cases}$$

Notation 3.1. We will denote by $L_n^{p,q}$ the qth level of the pth tower and by A_n^p its base, where $1 \le p \le n$ and $1 \le q \le n-p+1$. By definition, we will let $L_n^{p,n-p+1} = A_n^p$.

Definition 3.2. When $(x, y, w) \in C_n^3$, we define the trees inductively as follows:

$n=1$:

$A_1^1[x, y, w]$ is the unique thin filler of

$(T_2^1[y, w], -, T_2^1[x, yw], T_2^1[x, y]) \in \Lambda_2^1$.

Using [14, axiom T2], we may define thin elements by giving the horns that they fill, which we now proceed to do.

First tower

$n \ge 2$:

$$d_i L_n^{1,1}[x, y, w] = \begin{cases} T_{n+1}^1[y, w], & i = 0, \\ T_{n+1}^1[xy, w], & i = 1, \\ L_{n-1}^{1,1}[x_{n-1}, y_{n-1}, w_{i-2}], & 3 \le i \le n+2; \end{cases}$$

$n \ge 3$, $1 < q \le n-1$:

$$d_i L_n^{1,q}[x, y, w] = \begin{cases} T_{n+1}^q[y, w], & i = 0, \\ F_n^{1,q}[x, y, w], & i = 1, \\ L_{n-1}^{1,q-1}[x_{i-1}, y_{i-1}, w_{i-1}], & 2 \le i \le q-1, \\ d_q L_n^{1,q-1}[x, y, w], & i = q, \\ L_{n-1}^{1,q}[x_{n-1}, y_{n-1}, w_{i-2}], & q+2 \le i \le n+2; \end{cases}$$

First base

$n \ge 2$:

$$d_i A_n^1[x, y, w] = \begin{cases} T_{n+1}^n[y, w], & i = 0, \\ A_{n-1}^1[x_{i-1}, y_{i-1}, w_{i-1}], & 2 \le i \le n-1, \\ d_n L_n^{1,n-1}[x, y, w], & i = n, \\ T_{n+1}^1[x, yw], & i = n+1, \\ T_{n+1}^1[x, y], & i = n+2; \end{cases}$$

Intermediate towers

$n \geq 3,\ 1 < p \leq n-1$:

$$d_i L_n^{p,1}[x,y,w] = \begin{cases} L_{n-1}^{p-1,1}[x_i, y_i, w_i], & 0 \leq i \leq p-2, \\ d_{p-1} L_n^{p-1,2}[x,y,w], & i = p-1, \\ T_{n+1}^{p}[xy, w], & i = p, \\ L_{n-1}^{p,1}[x_{n-1}, y_{n-1}, w_{i-2}], & p+2 \leq i \leq n+2; \end{cases}$$

$n \geq 3,\ 1 < p \leq n-1,\ 1 < q \leq n-p$:

$$d_i L_n^{p,q}[x,y,w] = \begin{cases} L_{n-1}^{p-1,q}[x_i, y_i, w_i], & 0 \leq i \leq p-2, \\ d_{p-1} L_n^{p-1,q+1}[x,y,w], & i = p-1, \\ F_n^{p,q}[x,y,w], & i = p, \\ L_{n-1}^{p,q-1}[x_{i-1}, y_{i-1}, w_{i-1}], & p+1 \leq i \leq p+q-2, \\ d_{p+q-1} L_n^{p,q-1}[x,y,w], & i = p+q-1, \\ L_{n-1}^{p,q}[x_{n-1}, y_{n-1}, w_{i-2}], & p+q+1 \leq i \leq n+2; \end{cases}$$

Intermediate bases

$n \geq 3,\ 1 < p < n$:

$$d_i A_n^{p}[x,y,w] = \begin{cases} A_{n-1}^{p-1}[x_i, y_i, w_i], & 0 \leq i \leq p-2, \\ d_{p-1} A_n^{p-1}[x,y,w], & i = p-1, \\ A_{n-1}^{p}[x_{i-1}, y_{i-1}, w_{i-1}], & p+1 \leq i \leq n-1, \\ d_n L_n^{p,n-p}[x,y,w], & i = n, \\ T_{n+1}^{p}[x, yw], & i = n+1, \\ T_{n+1}^{p}[x,y], & i = n+2; \end{cases}$$

Last base (*root*)

$n \geq 2$:

$$d_i A_n^{n}[x,y,w] = \begin{cases} A_{n-1}^{n-1}[x_i, y_i, w_i], & 0 \leq i \leq n-2, \\ d_{n-1} A_n^{n-1}[x,y,w], & i = n-1, \\ T_{n+1}^{n}[x, yw], & i = n+1, \\ T_{n+1}^{n}[x,y], & i = n+2, \end{cases}$$

where the $F_n^{p,q}$ are constructed as in Scholium (3.0).

Proof of the associativity

Proposition 3.3. $d_n A_n^n[x,y,w] = T_{n+1}^n[xy, w]$.

Proof. We show that each fills the same n-horn.

$0 \leq i \leq n-2$:

$$d_i d_n A_n^n = d_{n-1} d_i A_n^n = d_{n-1} A_{n-1}^{n-1}[x_i, y_i, w_i]$$
$$= T_n^{n-1}[x_i y_i, w_i] = d_i T_{n+1}^n[xy, w];$$

$i = n-1$:

$$d_{n-1} d_n A_n^n = d_{n-1} d_{n-1} A_n^n = d_{n-1} d_{n-1} A_n^{n-1} = d_{n-1} d_n A_n^{n-1}$$
$$= d_{n-1} d_n L_n^{n-1,1} = d_{n-1} d_{n-1} L_n^{n-1,1} = d_{n-1} T_{n+1}^{n-1}[xy, w]$$
$$= d_{n-1} T_{n+1}^n[xy, w];$$

$i = n+1$:

$$d_{n+1} d_n A_n^n = d_n d_{n+2} A_n^n = d_n T_{n+1}^n[x, y] = xy = d_{n+1} T_{n+1}^n[xy, w]. \qquad \square$$

Corollary 3.4.

$$x(yw) = d_n T_{n+1}^n[x, yw] = d_n d_{n+1} A_n^n = d_n d_n A_n^n = d_n T_{n+1}^n[xy, w] = (xy)w. \qquad \square$$

4. Identities

We will now show that $(s_0^n d_1 d_0^{n-1} x)x = x$ and $x(s_0^n d_0^n x) = x$.

Lemma 4.0. *Suppose* $x \in X_n$ $(n \geq 1)$ *with* $d_n x = s_0^{n-1} a$, *where* $a \in X_0$. *Then* $T_{n+1}^j[s_0^n d_1 d_0^{n-1} x, x] = s_{j-1} x$. $\square$

Corollary 4.1.

$$T_{n+1}^n[s_0^n d_1 d_0^{n-1} x, x] = s_{n-1} x$$

implies

$$(s_0^n d_1 d_0^{n-1} x)x = x. \qquad \square$$

Lemma 4.2. *Suppose* $x \in X_n$ $(n \geq 1)$ *with* $d_n x = s_0^{n-1} a$, *where* $a \in X_0$. *Then* $T_{n+1}^j[x, s_0^n d_0^n x] = s_j^{n+1-j} d_j^{n-j} x$. $\square$

Corollary 4.3.

$$T_{n+1}^n[x, s_0^n d_0^n x] = s_n x$$

implies

$$x(s_0^n d_0^n x) = x. \qquad \square$$

5. Inverses

Lemma 5.0. *We will construct chains of horns inductively, which determine inverses.*

$n=1$.

When $x\in X_1$, let $I_1^1[x]$ be the unique thin filler of $(—, s_0d_1x, x)$.

$n>1$.

If $x\in X_n$, with $d_nx=s_0^{n-1}a$ where $a\in X_0$, then there exists a unique $I_n^j\in T_{n+1}$ such that

$j=1$:

$$d_iI_n^1[x]=\begin{cases} I_{n-1}^1[x_i], & 0\le i\le n-2,\\ s_0^n a, & i=n,\\ x, & i=n+1;\end{cases}$$

$1<j\le n$:

$$d_iI_n^j[x]=\begin{cases} I_{n-1}^j[x_i], & 0\le i\le n-j-1,\\ d_{n-j+1}I_n^{j-1}[x], & i=n-j+1,\\ I_{n-1}^{j-1}[x_{i-1}], & n-j+2\le i\le n,\\ T_n^{n-j+1}[x_{n-1}, s_0^{n-1}b], & i=n+1,\end{cases}$$

where $b=\operatorname{cod}x=d_0^nx$. □

Observe $\operatorname{dom}x=d_1d_0^{n-1}x=d_0^{n-1}d_nx=d_0^{n-1}s_0^{n-1}a=a$.

Definition 5.1. Let $x^{-1}=d_0I_n^n[x]$ for $n\ge 1$.

Proposition 5.2. $T_{n+1}^j[x,x^{-1}]=I_n^{n+1-j}[x]$ *for* $1\le j\le n$, $n\ge 1$. □

Observe that when $x\in X_n$ with $d_nx=s_0^{n-1}a$ $(a\in X_0)$, x^{-1} has shell $(x_0^{-1}, x_1^{-1},\dots, x_{n-1}^{-1}, s_0^{n-1}d_0^nx)$. Therefore xx^{-1} and $x^{-1}x$ are defined.

$$T_{n+1}^n[x,x^{-1}]=I_n^1[x]$$

implies

$$xx^{-1}=d_nT_{n+1}^n[x,x^{-1}]=d_nI_n^1[x]=s_0^na=s_0^n(d_1d_0^{n-1}x).$$

Thus x^{-1} is indeed the inverse of x.

So when $(X,T)\in T$-**Set**, if we let $G_{n-1}=\{x\in X_n \mid d_nx=s_0^{n-1}a, \text{ where } a\in X_0\}$ for $n\ge 1$, we have shown

$$G_{n-1}\ \underset{\operatorname{dom}=d_1d_0^{n-1}}{\overset{\operatorname{cod}=d_0^n}{\rightrightarrows}}\ X_0$$

is a groupoid where

$$xy=d_nT_{n+1}^n[x,y],\quad\text{and}\quad x^{-1}=d_0I_n^n[x].$$

6. The equivalence of $(m-1)$-Jgpd($\mathbb{C}$) and m-T-$\mathbb{C}$

First we show $\bar{W}G\simeq 1$.

When $X._m \in m\text{-}T\text{-}\mathbb{C}$, let $X_n = (X._m)_n$. Let $\tilde{X}_n = \{x \in X_n \mid d_n x = s_0^{n-1} a$ for some $a \in X_0\}$ for $n \geq 2$ and when $n = 1$, let $\tilde{X}_1 = X_1$.

Unraveling the definitions of G and $\bar{W}$, we have

$$(\bar{W}G(X._m))_0 = X_0, \qquad (\bar{W}G(X._m))_1 = X_1,$$

and for $n \geq 2$,

$$(\bar{W}G(X._m))_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \tilde{X}_i, d_0^{j-1} x_{j-1} = d_1 d_0^{j-2} d_{j-1} x_j,\ 2 \leq j \leq n\}.$$

Scholium 6.0. Let T_k be the thin filler of the k-horn

$$(h_0, \ldots, h_{k-2}, h_{k-1}, h_{k+1}, \ldots, h_n).$$

Let T be the thin filler of the $(k-1)$-horn

$$(h_0, \ldots, h_{k-2}, d_k T_k, h_{k+1}, \ldots, h_n).$$

Then T_k fills the $(k-1)$-horn as well, hence $T = T_k$.

Therefore $d_{k-1} T = d_{k-1} T_k = h_{k-1}$.

Proposition 6.1 (The folding operation). *Let*

$$\mathscr{C}_n = \{(x, y) \in X_{n-1} \times \tilde{X}_n \mid d_0^{n-1} x = d_1 d_0^{n-2} d_{n-1} y\} \quad \textit{for } n \geq 2.$$

If $(x, y) \in \mathscr{C}_n$, *then* $\varphi_n(x, y) = T_n^{n-1}[x, d_n y]\, y \in X_n$.

Conversely, given $w \in X_n$, $\exists!(x, y) \in \mathscr{C}_n$ *such that* $\varphi_n(x, y) = w$.

Proof. $n = 2$ (existence). Recall the definition of zy where $(z, y) \in C_2^2$:

$$\begin{aligned}
&\text{thin } T_3^1[z, y] \quad \text{fills} \quad (y, —, T_2^1[z_1, y_1], T_2^1[z_1, y_2]);\\
&\text{thin } T_3^2[z, y] \quad \text{fills} \quad (T_2^1[z_0, y_0], d_1 T_3^1[z, y], —, z);\\
&zy = d_2 T_3^2[z, y].
\end{aligned}$$

Given $w \in X_2$, let $x = d_2 w$, $a = d_0 x$, and $z = T_2^1[x, s_0 a]$.

Let $y_0 = d_0 w$, $y_1 = x^{-1} d_1 w$, $y_2 = s_0 a$. Let

$$\text{thin } F_2 \quad \text{fill} \quad (T_2^1[z_0, y_0], —, w, z)$$

and

$$\text{thin } F_1 \quad \text{fill} \quad (—, d_1 F_2, T_2^1[z_1, y_1], T_2^1[z_1, y_2]).$$

Let $y = d_0 F_1$.

Scholium 6.0 implies

$$T_3^1[z, y] = F_1,$$

so $d_1 T_3^1[z, y] = d_1 F_1 = d_1 F_2$.

Scholium 6.0 implies

$$T_3^2[z, y] = F_2,$$

so $\varphi_2(x, y) = d_2 T_3^2[z, y] = d_2 F_2 = w$.

$n=2$ (uniqueness).

Suppose $\varphi_2(x, y) = w$.

$(x, y) \in \mathscr{C}_2$ implies that there exists $a \in X_0$ such that $d_2 y = s_0 a$ and $d_0 x = d_1 d_1 y$, so $d_0 x = d_1 d_2 y = d_1 s_0 a = a$.

$$T_2^1[x, s_0 a]\, y = w$$

implies

$$x = d_2 w, \qquad d_0 y = d_0 w, \quad \text{and} \quad d_1 y = x^{-1} d_1 w.$$

Thus a, x, $d_0 y$, $d_1 y$, and $d_2 y$ are determined by w.

Let $z = T_2^1[x, s_0 a]$ (so $zy = w$).

Scholium 6.0 implies

$$F_2 = T_3^2[z, y],$$

so $d_1 F_2 = d_1 T_3^2[z, y] = d_1 T_3^1[z, y]$.

Scholium 6.0 implies

$$F_1 = T_3^1[z, y],$$

so $d_0 F_1 = y$.

Hence x and y are determined by w.

When $n > 2$ the proof is similar. □

Corollary 6.2.

$$(\bar{W}G(X_{\cdot m}))_n \stackrel{\sim}{\to} X_n,$$
$$(x_1, x_2, \ldots, x_n) \mapsto \varphi_n(\cdots \varphi_3(\varphi_2(x_1, x_2), x_3) \cdots, x_n). \qquad \square$$

Now we show that $G\bar{W} \simeq 1$.

Let

$$\begin{array}{cccc}
 & & & \vdots \\
 & G_3 & \rightrightarrows & O \\
 & \downarrow\downarrow\downarrow\downarrow & & \| \\
 & G_2 & \rightrightarrows & O \\
\mathscr{G}: & \downarrow\downarrow\downarrow & & \| \\
 & G_1 & \rightrightarrows & O \\
 & \downarrow\downarrow & & \| \\
 & G_0 & \rightrightarrows & O
\end{array}$$

be in $(m-1)$-**Jgpd**($\mathbb{C}$).

Let $a_{i,0} \in O$ and $a_{i,j} \in G_i^j = (\mathrm{Ner}(G_i))_j$.

Let $a = a_{0,0} \in O$ and $(\bar{W}(\mathscr{G}))_{j-1} \xrightarrow{s_0} (\bar{W}(\mathscr{G}))_j$.

$$\begin{aligned}
s_0 a &= (1_a, a),\\
s_0^2 a &= s_0(1_a, a) = ((1_a, 1_a), 1_a, a),\\
&\vdots\\
s_0^{n-1} a &= (\underbrace{(1_a, \dots, 1_a)}_{0}^{n-1}, \underbrace{(1_a, \dots, 1_a)}_{1}^{n-2}, \dots, \underset{n-2}{1_a}, \underset{n-1}{a}).
\end{aligned}$$

Let $(a_{0,n}, a_{1,n-1}, \dots, a_{n,0}) \in (\bar{W}(\mathscr{G}))_n$.

$d_n(a_{0,n}, a_{1,n-1}, \dots, a_{n,0}) = s_0^{n-1} a$ implies the following:

$$\begin{aligned}
a_{0,n} &= (1_a, \dots, 1_a, b_{0,n}),\\
a_{1,n} &= (1_a, \dots, 1_a, b_{1,n-1}),\\
&\vdots\\
a_{n-2,2} &= (1_a, b_{n-2,2}),\\
d_1^H a_{n-1,1} &= a,
\end{aligned}$$

where $b_{p,q} \in G_p^q$.

$d_0^H(a_{p,n-p}) = d_{p+1}^V(a_{p+1,n-p-1})$ for $0 \le p \le n-1$ implies the following:

$$\begin{aligned}
&d_j^V b_{j,n-j} = b_{j-1,n-j+1} \quad \text{where } 1 \le j \le n-2,\\
&d_{n-1}^V a_{n-1,1} = b_{n-2,2}, \qquad d_0^H a_{n-1,1} = d_n^V a_{n,0} = a_{n,0}.
\end{aligned}$$

Thus $a_{n-1,1} \in G_{n-1}$ determines $(a_{0,n}, a_{1,n-1}, \dots, a_{n,0})$ with $d_n(a_{0,n}, a_{1,n}, \dots, a_{n,0}) = s_0^{n-1} a$.

Hence, $(G\bar{W}(\mathscr{G}))_{n-1} \simeq G_{n-1}$ in $\mathbb{C}$.

When $g \in G_{n-1}$, let $g' \in (\bar{W}(\mathscr{G}))_n$ be the element determined, with $d_n g' = s_0^{n-1} a$ for some $a \in (\bar{W}(\mathscr{G}))_0 = O$.

If $(a_{0,m}, a_{1,m}, \dots, a_{m-1,1}, a_{m,0}) \in (\bar{W}(\mathscr{G}))_m$, then

(i) $a_{m,0} = d_0^H a_{m-1,1} \in O$;

(ii) $a_{p,m-p} \in (\mathrm{Ner}(G_p))_{m-p}$, $0 \le p \le m-1$;

(iii) $d_0^H(a_{p,m-p}) = d_{p+1}^V(a_{p+1,m-p-1})$, $0 \le p \le m-1$.

So such an element would be determined by $a_{m-1,1} \in G_{m-1}$ and $a_{p,m-p}^1 \in G_p$ where $0 \le p \le m-2$. ($a_{p,m-p}^1$ is the first arrow of a composable sequence of arrows in G_p of length $m-p$.)

When $x, y \in G_{n-1}$ with $\mathrm{cod}\, x = \mathrm{dom}\, y$, we define $t_{n+1}^j \in (\bar{W}(\mathscr{G}))_{n+1}$ for $1 \le j \le n$ as follows:

$$\begin{aligned}
&(t_{n+1}^j)_n = s_{j-1}^V y;\\
&(t_{n+1}^j)_{j-1}^1 = \begin{cases} d_j^V d_{j+1}^V \cdots d_{n-1}^V(x), & 1 \le j \le n-1,\\ x, & j = n;\end{cases}\\
&(t_{n+1}^j)_p^1 = 1, \quad p \ne j-1.
\end{aligned}$$

Observe $(t^n_{n+1})_{n-1}=(x,y)\in G^2_{n-1}$.

A straightforward calculation shows $t^j_{n+1}=T^j_{n+1}[x',y']\in(\bar{W}(\mathscr{G}))_{n+1}$.

We now see that the groupoid structures on $(G\bar{W}(\mathscr{G}))_{n-1}$ and G_{n-1} are the same because

$$\begin{aligned}(x'y')_{n-1} &= (d_n T^n_{n+1}[x',y'])_{n-1} = (d_n t^n_{n+1})_{n-1}\\ &= d^H_1((t^n_{n+1})_{n-1,2}) = d^H_1(x,y) = xy.\end{aligned}$$

Thus $(G\bar{W}(\mathscr{G}))_{n-1}\simeq G_{n-1}$.

7. The equivalence n-Gpd($\mathbb{C}$)$\approx n$-T-$\mathbb{C}$

Recall that G is an n-groupoid in $\mathbb{C}$ if $G_n \rightrightarrows G_{n-1}$ is a groupoid object in $(n-1)$-**Gpd**($\mathbb{C}$) with

$$\begin{array}{ccc} G_n & \rightrightarrows & G_{n-1} \\ \downdownarrows & & \downdownarrows \\ G_{n-2} & = & G_{n-2} \end{array}$$

when viewed in $(n-2)$-**Gpd**($\mathbb{C}$). We will say G is an *n-groupoid over* G_{n-1}.

Given $(X_{\cdot 1}\hookrightarrow X_{\cdot 2}\hookrightarrow\cdots\hookrightarrow X_{\cdot m})\in m$-**Jgpd**($\mathbb{C}$), let $O=\mathrm{Ob}(X_{\cdot m})_0\in\mathbb{C}$. We will say $X_{\cdot m}$ is an *m-Jgroupoid over O*.

When G is an n-groupoid in $\mathbb{C}$ observe that:

> m-T-complexes in the category of $(n-m)$-groupoids over G_{n-m-1} may be viewed as m-Jgroupoids over G_{n-m-1} in the category of $(n-m-1)$-groupoids over G_{n-m-2} which are naturally equivalent to $(m+1)$-T-complexes in the category of $(n-m-1)$-groupoids over G_{n-m-2}, for $m=2,\dots,n-1$.

From which it follows that n-**Gpd**($\mathbb{C}$)$\approx n$-T-$\mathbb{C}$.

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